

On convergence of the immersed boundary method for elliptic interface problems

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Abstract

Peskin's Immersed Boundary (IB) method is one of the most popular numerical methods for many years and has been applied to problems in mathematical biology, fluid mechanics, material sciences, and many other areas. Peskin's IB method is associated with discrete delta functions. It is believed that the IB method is first order accurate in the L^∞ norm. But almost none rigorous proof could be found in the literature until recently [9] in which the author showed that the velocity is indeed first order accurate for the Stokes equations with a periodic boundary condition. In this paper, we show the first order convergence with a $\log h$ factor of the IB method for elliptic interface problems essential without the boundary condition restrictions. The results should be applicable to the IB method for many different situations involving elliptic solvers for Stokes and Navier-Stokes equations.

keywords: Immersed Boundary (IB) method, Dirac delta function, convergence of IB method, discrete Green function, discrete Green's formula.

AMS Subject Classification 2000 65M12, 65M20.

1 Introduction

Since its invention in 1970's, the Immersed Boundary (IB) method [10] has been applied to almost everywhere in mathematics, engineering, biology, fluid mechanics, and many many more areas, see for example, [11] for a review and references therein. The IB method is not only a mathematical modeling tool in which a complicated boundary condition can be treated as a source distribution but also a numerical method in which a discrete delta function is used. The IB method is robust, simple, and has been applied to many problems.

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14. ABSTRACT Peskin's Immersed Boundary (IB) method is one of the most popular numerical methods for many years and has been applied to problems in mathematical biology, fluid mechanics material sciences, and many other areas. Peskin's IB method is associated with discrete delta functions. It is believed that the IB method is first order accurate in the L1 norm. But almost none rigorous proof could be found in the literature until recently [9] in which the author showed that the velocity is indeed first order accurate for the Stokes equations with a periodic boundary condition. In this paper, we show the first order convergence with a log h factor of the IB method for elliptic interface problems essential without the boundary condition restrictions. The results should be applicable to the IB method for many different situations involving elliptic solvers for Stokes and Navier-Stokes equations.					
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It is widely believed that Peskin's IB method is only first order accurate in the L^∞ norm. However, there was almost none rigorous proof in the literature until recently [9], in which the author has proved the first order accuracy of the IB method for the Stokes equations with a periodic boundary condition, see also [3] in which the author showed that the pressure obtained from IB method has $O(h^{1/2})$ order of convergence in the L^2 norm for a 1D model. However, there are few theoretical proof on the IB method for elliptic interface problems. This is the main motivation of this section. The main goal of this paper is to provide a convergence proof for the IB method. We will show that with reasonable assumptions (zero-th moment condition and first order interpolation property) on the discrete delta function used in the IB method, the IB method is indeed first order convergent in the L^∞ norm. Our proof is essentially independent of the boundary conditions and it is valid in 1D, 2D, and 3D cases. The result should be applicable for many IB methods involving Stokes and Navier-Stokes solvers.

2 Proof of the convergence of the IB method in 1D

We will give a proof for the 1D model first,

$$u'' = c\delta(x - \alpha) \quad 0 < x < 1, \quad 0 < \alpha < 1, \quad u(0) = u(1) = 0, \quad (1)$$

in this section. Note that the analytic solution to the equation above is

$$u(x) = \begin{cases} -cx(1 - \alpha) & \text{if } x \leq \alpha, \\ -c\alpha(1 - x) & \text{otherwise.} \end{cases} \quad (2)$$

Given a uniform Cartesian grid $x_i = ih$, $i = 0, 1, \dots, n$, $h = 1/n$, the IB method leads to the following system of linear equations,

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = c\delta_h(x_i - \alpha), \quad i = 1, 2, \dots, n-1, \quad (3)$$

where U_i is the finite difference approximation of the solution $u(x_i)$, and $\delta_h(x_i - \alpha)$ is a discrete delta function applied to the grid point x_i . In the matrix-vector form, the above finite difference equations can be written as $\mathbf{A}\mathbf{U} = \mathbf{F}$, where \mathbf{A} is the tri-diagonal matrix formed by the discrete Laplacian. It is well known that $-\mathbf{A}$ is a symmetric positive definite matrix and diagonally dominant. Note that, a discrete delta function has a compact support in the neighborhood of interface, that is, $\delta_h(x) \neq 0$ only if $|x| \leq Wh$, where W is a constant. Commonly used discrete delta functions include the hat discrete delta function ($\delta^{hat}(x)$ with $W = 1$) and the cosine discrete delta functions ($\delta^{cosine}(x)$ with $W = 2$), see for example, [8]. Note that, when we use the hat delta function, the result is the same as that of the IIM for the simple model. The solution to the finite difference equations is the same as true solution if there are no round-off errors, that is

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1}))}{h^2} = c\delta_h^{hat}(x_i - \alpha), \quad i = 1, 2, \dots, n-1, \quad (4)$$

see for example, [1, 8]. But this is not the case for other discrete delta functions.

We define the error vector as $\mathbf{E} = \{E_i\}$, where $E_i = u(x_i) - U_i$. The local truncation error is defined as $\mathbf{T} = \{T_i\}$,

$$T_i = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1}))}{h^2} - c\delta_h(x_i - \alpha). \quad (5)$$

With the definition, we have $A_h \mathbf{U} = \mathbf{F}$, $A_h \mathbf{u} = \mathbf{F} + \mathbf{T}$, and therefore $A_h \mathbf{E} = \mathbf{T}$. For the hat discrete delta function, we have $|T_i| = 0$ for all i 's. For other discrete delta functions, we have $|T_j| = O(1/h)$ for a few grid points neighboring the interface α . So the interesting question is: Why is the IB method still first order accurate, that is, $\|\mathbf{E}\|_\infty = O(h)$? To answer this question, we first introduce the following lemma.

Lemma 2.1. *Let $A_h \mathbf{y} = \mathbf{e}_k$, where \mathbf{e}_k is the k -th unit base vector, $y_0 = y_n = 0$, then*

$$y_i = \begin{cases} -hx_i(1 - x_k) & \text{if } i \leq k, \\ -hx_k(1 - x_i) & \text{otherwise.} \end{cases} \quad (6)$$

The significance of this lemma is that the solution is order h smaller than the concentrated source.

Proof: We note the following identity

$$A_h \mathbf{y} = h \frac{\mathbf{e}_k}{h} = h \delta_h^{hat} \mathbf{e}_k. \quad (7)$$

For this simple case, the IB and IIM are identical. Thus from the Immersed Interface Method, see [1, 8], we know that \mathbf{y} is the exact discrete solution at the grid points of the following boundary value problem

$$u'' = h \delta(x - x_k), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad (8)$$

whose solution is

$$y(x) = \begin{cases} -hx(1 - x_k) & \text{if } x \leq x_k, \\ -hx_k(1 - x) & \text{otherwise.} \end{cases} \quad (9)$$

This completes the proof. Note that $|y(x)| \leq h$. From this lemma, we have the following corollary.

Corollary 2.2. *Let $A_h \mathbf{y} = \mathbf{r}$, then $|y(x)| \leq hW$, where W is the number of non-zero components of \mathbf{r} .*

Notice that for a discrete delta function, it should satisfy at least the zeroth moment equation, see [2], that is

$$\sum_i \delta_h(x_i - \alpha) = 1, \quad (10)$$

corresponding to the continuous case $\int \delta_h(x - \alpha) dx = 1$. Now we are ready to prove the main theorem.

Theorem 2.3. *The Immersed Boundary method for (1) is first order accurate, that is*

$$\|\mathbf{E}\|_\infty \leq \bar{C}h, \quad (11)$$

where \bar{C} is a constant.

Proof: We can decompose the local truncation error into two groups

$$\mathbf{T} = \mathbf{T}^{reg} + \mathbf{T}^{irreg}, \quad (12)$$

where $\|\mathbf{T}^{reg}\|_\infty = 0$ corresponds to the local truncation errors at regular grid points where $\delta_h(x_i - \alpha) = 0$ and the true solution is piecewise linear. Note that, we have

$$\sum_{i=1}^{n-1} T_i = \sum \mathbf{T}^{reg} + \sum \mathbf{T}^{irreg} = O + \sum \mathbf{T}^{irreg}. \quad (13)$$

On the other hand, we also have

$$\sum_{i=1}^{n-1} T_i = \sum_{i=1}^{n-1} c\delta_h^{hat}(x_i - \alpha) - \sum_{i=1}^{n-1} c\delta_h(x_i - \alpha) = 0, \quad (14)$$

since the finite difference method using the discrete delta function gives the exact solution at the grid points. Thus we have $\sum_i T_i = T_i^{irreg} = 0$. We divide T_i^{irreg} into two groups, one with all positive T_i 's denoted as $T_i^{irreg,+}$; the other one is all the negatives denoted as $T_i^{irreg,-}$. Since $\sum T_i^{irreg,+} + \sum T_i^{irreg,-} = 0$, $T_i^{irreg,+}$ and $T_i^{irreg,-}$ must have the same order of the magnitude ($O(1/h)$) although those index i are different except that $|x_i - \alpha| \leq Wh$ is true for all irregular grid points. Because the solution is linear with c , we have

$$\mathbf{E} = A_h^{-1} \mathbf{T} = A_h^{-1} (\mathbf{T}^{irreg,+} + \mathbf{T}^{irreg,-}). \quad (15)$$

From the solution expression, we know that, assuming that $x_k \leq \alpha$,

$$\begin{aligned} E_k &= -x_k \left(\sum_i T_i^{irreg,+} (1 - x_i) + \sum_j T_j^{irreg,-} (1 - x_j) \right) \\ &= -x_k (1 - \alpha) \left(\sum_i T_i^{irreg,+} + \sum_j T_j^{irreg,-} \right) + O(Wh) \\ &= O(Wh), \end{aligned}$$

after we expand all x_i 's and x_k 's at α and since all related x_i and x_j are within Wh distance from the interface α . The proof when $x_k > \alpha$ is similar except that we need to use the solution for $x > \alpha$. This completes the proof.

3 Proof of the convergence of the IB method in 2D

The discussion for 2D problems is much more challenging since the interface is often a curve instead of a point. In [9], the author has proved the first order accuracy of the IB method for the Stokes equations with a periodic boundary condition in 2D. However, there are almost no theoretical proof on the IB method for elliptic interface problems or other general boundary conditions. We will prove that the result obtained from the IB for the elliptic interface problem is indeed first order accurate in 2D as well in this section.

Consider the 2D elliptic interface problem

$$\begin{aligned}\Delta u(x, y) &= f(x, y) + \int_{\Gamma} v(s) \delta(x - X(s)) (y - Y(s)) ds \\ u(x, y)|_{\partial\Omega} &= u_0(x, y),\end{aligned}\tag{16}$$

where we assume that $f \in C(\Omega)$, $\Gamma \in C$, $v(s) \in C$. Without loss of generality, we assume that Ω is a unite square $0 \leq x, y \leq 1$. The problem can be decomposed as the sum of the solutions of the following two problems. The first one is

$$\begin{aligned}\Delta u_1(x, y) &= f(x, y) \\ u_1(x, y)|_{\partial\Omega} &= u_0(x, y),\end{aligned}\tag{17}$$

which is a regular problem whose solution $u_1(x, y) \in C^2(\Omega)$. The second problem is

$$\begin{aligned}\Delta u_2(x, y) &= \int_{\Gamma} v(s) \delta(x - X(s)) (y - Y(s)) ds \\ u_2(x, y)|_{\partial\Omega} &= 0,\end{aligned}\tag{18}$$

The solution to the second problem is equivalent to the following problem

$$\begin{aligned}\Delta u_2(x, y) &= 0, \quad [u_2]_{\Gamma} = 0, \quad \left[\frac{\partial u_2}{\partial n} \right]_{\Gamma} = v(s), \\ u_2(x, y)|_{\partial\Omega} &= 0.\end{aligned}\tag{19}$$

The solution to the original problem is $u = u_1 + u_2$. Since u_1 is the solution to a regular problem, it is enough just to consider $u_2(x, y)$, and we will simply use the notation $u(x, y)$ for $u_2(x, y)$.

The Peskin's IB method for the problem including the following steps.

- Generating a uniform Cartesian mesh $x_i = ih$, $y_j = jh$, $i, j = 0, 1, \dots, n$.
- Replace the partial derivatives with the finite difference approximation and use a discrete delta function

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} = C_{ij}^{IB}, \quad i, j = 0, 1, \dots, n,\tag{20}$$

$$C_{ij}^{IB} = \sum_{k=1}^{N_b} v_k \delta_h(x_i - X_k) \delta_h(y_j - Y_k) \Delta s_k,\tag{21}$$

where (X_k, Y_k) , $k = 1, 2, \dots, N_b$ is a discretization of the interface Γ , and $v_k \approx v(s_k)$.

- Solve the finite difference system of equations above to get an approximation solution $\{U_{ij}\}$.

3.1 Assumptions and the discrete delta function

To guarantee enough accuracy, we assume that $\max_k \{s_k\} = \Delta s \sim O(h)$, and

$$\sum_{i,j=1}^{n-1} h^2 \sum_{k=1}^{N_b} v_k \delta_h(x_i - X_k) \delta_h(y_j - Y_k) \Delta s_k = \int_{\Gamma} v(s) ds + O(h), \quad (22)$$

see for example, [9], which corresponds to

$$\iint_{\Omega} \int_{\Gamma} v(s) \delta(x - X(s)) \delta(y - Y(s)) ds dx dy = \int_{\Gamma} v(s) ds. \quad (23)$$

From $\iint_{\Omega} u(x, y) \delta(x - X) \delta(y - Y) dx dy = u(X, Y)$, we should also have the interpolation property,

$$\sum_{i,j=1}^{n-1} h^2 u(x_i, y_j) \delta_h(x_i - X_k) \delta_h(y_j - Y_k) = u(X, Y) + O(h). \quad (24)$$

The discrete delta function has a compact support, that is,

$$\delta_h(x_i - X_k) = 0, \quad \text{if } |x_i - X_k| > Wh, \quad \text{and} \quad \delta_h(y_j - Y_k) = 0, \quad \text{if } |y_j - Y_k| > Wh, \quad (25)$$

where $\mathbf{x}_{ij} = (x_i, y_j)$ and W is a constant.

We define the error vector as $\mathbf{E} = \{E_{ij}\}$, where $E_{ij} = u(x_i, y_j) - U_{ij}$. The local truncation error is defined as $\mathbf{T} = \{T_{ij}\}$,

$$T_{ij} = \frac{u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j)}{h^2} - C_{ij}^{IB}. \quad (26)$$

In the matrix vector form, we have $A_h \mathbf{U} = \mathbf{F}$, $A_h \mathbf{u} = \mathbf{F} + \mathbf{T}$, and therefore $A_h \mathbf{E} = \mathbf{T}$, where A_h is the matrix formed by the discrete Laplacian. In general, we have $|T_{ij}| = O(1/h)$ for grid points neighboring the interface Γ except for the correction terms using the Immersed Interface Method (IIM) [6, 7, 8] for which we have $|T_{ij}^{IIM}| = O(h)$ at irregular grid points where the interface cuts through the standard five-point stencil.

So it is interesting that the local truncation errors can have order $O(1/h)$ at some grid points, but the global error is still of order $O(h)$. There has to be some kind of cancelations of the errors, which can be seen from our proof process.

Definition 3.1. Let \mathbf{e}_{ij} be the unit grid function whose values are zero at all grid points except at $\mathbf{x}_{ij} = (x_i, y_j)$, $e_{ij} = 1$. The discrete Green function value at \mathbf{x}_{lm} centered at \mathbf{x}_{ij} is defined as the solution of the following

$$\mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) = \left((A_h)^{-1} \mathbf{e}_{ij} \frac{1}{h^2} \right)_{lm}. \quad (27)$$

See for example, [1, 4, 5, 9] for more discussions about the discrete Green's functions. To prepare the convergence proof, we first list some lemmas that are used in the proof.

Lemma 3.2. *The discrete Green's formula and an error estimate. Assuming that the distance between Γ and $\partial\Omega$ is $O(1)$, that is, $\text{dist}(\Gamma, \partial\Omega) \sim O(1)$, then we have*

$$\sum_{ij} \Delta_h u(x_i, y_j) h^2 = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds + O(h) = \int_{\Gamma} v(s) ds + O(h), \quad (28)$$

where

$$\Delta_h u(x_i, y_j) = \frac{u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j)}{h^2}, \quad (29)$$

is the discrete Laplacian using the standard five-point stencil.

Proof: We first show the discrete Green's formula by expanding the summation. After cancellation of interior terms, only boundary terms are left. Thus, we get

$$\begin{aligned} \sum_{ij} \Delta_h u(x_i, y_j) h^2 &= \sum_{j=1}^{n-1} h \frac{u(x_0, y_j) - u(x_1, y_j)}{h} + \sum_{j=1}^{n-1} h \frac{u(x_n, y_j) - u(x_{n-1}, y_j)}{h} \\ &\quad + \sum_{i=1}^{n-1} h \frac{u(x_i, y_0) - u(x_i, y_1)}{h} + \sum_{i=1}^{n-1} h \frac{u(x_i, y_n) - u(x_i, y_{n-1})}{h} \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} ds + O(h). \end{aligned}$$

On the other hand, by integrating both sides of the partial differential equation (16) with $f = 0$ and $u_0 = 0$, we get

$$\iint_{\Omega} \Delta u dx dy = \iint_{\Omega} \left(\int_{\Gamma} v(s) \delta(x - X(s)) (y - Y(s)) ds \right) dx dy,$$

or equivalently,

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\Gamma} v(s) ds.$$

This completes the proof.

Remark 3.3. *The double integral $\iint_{\Omega} \Delta u dx dy$ can be divided into three parts*

$$\begin{aligned} \iint_{\Omega} \Delta u dx dy &= \iint_{\Omega_{\epsilon}^+} \Delta u dx dy + \iint_{\Omega_{\epsilon}^-} \Delta u dx dy + \iint_{\Omega_{\epsilon}} \Delta u dx dy \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} ds - \int_{\Gamma_{\epsilon}^+} \frac{\partial u}{\partial n} ds + \int_{\Gamma_{\epsilon}^+} \frac{\partial u}{\partial n} ds, \end{aligned}$$

see Fig. 1 for an illustration, see also [7]. As $\epsilon \rightarrow 0$, we have

$$\lim_{\epsilon \rightarrow 0} \iint_{\Omega_\epsilon} \Delta u dx dy = \int_{\Gamma} v(s) ds = \int_{\Gamma} \left[\frac{\partial u}{\partial n} \right] ds$$

from the partial differential equation (16) with $f = 0$ and $u_0 = 0$. Thus we get

$$\lim_{\epsilon \rightarrow 0} \iint_{\Omega_\epsilon} \Delta u dx dy = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds.$$

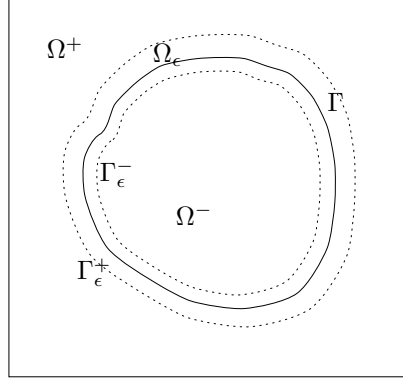


Figure 1: A diagram of the domain, interface, and integration.

Lemma 3.4. *There is a solution to the point source problem*

$$\Delta \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) = \delta(x - x_l)\delta(y - y_m), \quad \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm})|_{\partial\Omega} = 0. \quad (30)$$

Proof: We know that $u_1(\mathbf{x}) = \log|\mathbf{x} - \mathbf{x}_{lm}|/(2\pi)$ is a solution to $\Delta u_1(\mathbf{x}, \mathbf{x}_{lm}) = \delta(x - x_l)\delta(y - y_m)$. Let $u_2(\mathbf{x}, \mathbf{x}_{lm})$ be the solution to the following problem

$$\Delta u_2(\mathbf{x}, \mathbf{x}_{lm}) = 0, \quad u_2(\mathbf{x}, \mathbf{x}_{lm})|_{\partial\Omega} = u_1(\mathbf{x}, \mathbf{x}_{lm})|_{\partial\Omega}. \quad (31)$$

This is a regular problem and the solution u_2 is unique. Then $\tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) = u_1 - u_2$ is the solution to (30). Note that $\mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm})$ is a discrete approximation to $\tilde{\mathbf{G}}(\mathbf{x}_{ij}, \mathbf{x}_{lm})$. The Green function $\Delta \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm})$ is a little different from the standard Green function $\mathbf{G}(\mathbf{x}_{ij}, \mathbf{x}_{lm})$ for which the boundary condition is undefined and the solution can differ by a constant.

We also define an interpolation function for $\mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm})$, for example,

$$\mathbf{G}_I^h(\mathbf{x}(s), \mathbf{x}_{lm}) = \sum_{ij} h^2 \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) \delta_h(x_i - x) \delta_h(y_j - y). \quad (32)$$

Note that the coefficients of the interpolation are of $O(1)$. Other type interpolation schemes such as bi-linear interpolation will lead to the same conclusion.

By similar procedure in proving the discrete Green's formula, we can get the second discrete Green's formula.

Lemma 3.5. *Assuming that the distance between Γ and $\partial\Omega$ is $O(1)$, that is, $\text{dist}(\Gamma, \partial\Omega) \sim O(1)$, then we have,*

$$\sum_{ij} \Delta_h u(x_i, y_j) \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 = \int_{\Gamma} v(s) \tilde{\mathbf{G}}(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h).$$

Proof: Again, we show the second discrete Green's formula by expanding the summation. After cancellation of interior terms, only boundary terms and a source are left. Thus, we get

$$\begin{aligned} \sum_{ij} \Delta_h u(x_i, y_j) \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 &= \sum_{j=1}^{n-1} h \frac{u(x_0, y_j) - u(x_1, y_j)}{h} \mathbf{G}^h(\mathbf{x}_{1j}, \mathbf{x}_{lm}) \\ &+ \sum_{j=1}^{n-1} h \frac{u(x_n, y_j) - u(x_{n-1}, y_j)}{h} \mathbf{G}^h(\mathbf{x}_{n-1,j}, \mathbf{x}_{lm}) + \sum_{i=1}^{n-1} h \frac{u(x_i, y_0) - u(x_i, y_1)}{h} \mathbf{G}^h(\mathbf{x}_{i1}, \mathbf{x}_{lm}) \\ &+ \sum_{i=1}^{n-1} h \frac{u(x_i, y_n) - u(x_i, y_{n-1})}{h} \mathbf{G}^h(\mathbf{x}_{i,n-1}, \mathbf{x}_{lm}) - \sum_{j=1}^{n-1} h \frac{\mathbf{G}^h(\mathbf{x}_{0j}, \mathbf{x}_{lm}) - \mathbf{G}^h(\mathbf{x}_{1j}, \mathbf{x}_{lm})}{h} u(x_1, y_j) \\ &- \sum_{j=1}^{n-1} h \frac{\mathbf{G}^h(\mathbf{x}_{n,j}, \mathbf{x}_{lm}) - \mathbf{G}^h(\mathbf{x}_{n-1,j}, \mathbf{x}_{lm})}{h} u(x_{n-1}, y_j) - \sum_{i=1}^{n-1} h \frac{\mathbf{G}^h(\mathbf{x}_{i0}, \mathbf{x}_{lm}) - \mathbf{G}^h(\mathbf{x}_{i1}, \mathbf{x}_{lm})}{h} u(x_i, y_1) \\ &- \sum_{i=1}^{n-1} h \frac{\mathbf{G}^h(\mathbf{x}_{i,n}, \mathbf{x}_{lm}) - \mathbf{G}^h(\mathbf{x}_{i,n-1}, \mathbf{x}_{lm})}{h} u(x_i, y_{n-1}) + u(x_l, y_m) \\ &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) - \frac{\partial \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm})}{\partial n} u \right) ds + u(x_l, y_m) + O(h). \end{aligned}$$

This is because $|\mathbf{G}^h(\mathbf{x}_{i,j}, \mathbf{x}_{lm}) - \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm})| \leq Ch/\text{dist}(\mathbf{x}_{ij}, \mathbf{x}_{lm})$ for boundary points \mathbf{x}_{ij} when \mathbf{x}_{lm} is near or on the interface Γ and far away from the boundary $\partial\Omega$, where C is a constant, see for example, [1, 4, 5]. On the other hand, by integrating both sides of the partial differential equation (16) with $f = 0$ and $u_0 = 0$, we get

$$\iint_{\Omega} \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) \Delta u dx dy = \iint_{\Omega} \left(\int_{\Gamma} v(s) \delta(x - X(s)) (y - Y(s)) ds \right) \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) dx dy,$$

or equivalently,

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) - \frac{\partial \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm})}{\partial n} u \right) ds + \iint_{\Omega} u \Delta \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) dx dy = \int_{\Gamma} v(s) \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) ds.$$

Note that

$$\iint_{\Omega} u \Delta \tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}_{lm}) dx dy = \iint_{\Omega} u(x, y) \delta(x - x_l) \delta(y - y_m) dx dy = u(x_l, y_m).$$

This completes the proof.

Lemma 3.6.

$$\sum_{ij} C_{ij}^{IB} \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 = \int_{\Gamma} v(s) \tilde{\mathbf{G}}(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h \log h).$$

Proof:

$$\begin{aligned} \sum_{ij} C_{ij}^{IB} \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 &= \sum_{ij} \sum_{k=1}^{N_b} v_k \delta_h(x_i - X_k) \delta_h(y_j - Y_k) \Delta s_k \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 \\ &= \sum_{k=1}^{N_b} v_k \Delta s_k \sum_{ij} \delta_h(x_i - x) \delta_h(y_j - y) \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) h^2 \\ &= \sum_{k=1}^{N_b} v_k \Delta s_k \mathbf{G}_I^h(\mathbf{X}_k, \mathbf{x}_{lm}) \\ &= \sum_{|\mathbf{X}_k - \mathbf{x}_{lm}| > Wh} v_k \Delta s_k \mathbf{G}_I^h(\mathbf{X}_k, \mathbf{x}_{lm}) + \sum_{|\mathbf{X}_k - \mathbf{x}_{lm}| \leq Wh} v_k \Delta s_k \mathbf{G}_I^h(\mathbf{X}_k, \mathbf{x}_{lm}) \\ &= \int_{\Gamma - \Gamma_{\mathbf{x}_{lm}}^h} v(s) \mathbf{G}_I^h(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h) + \int_{\Gamma_{\mathbf{x}_{lm}}^h} v(s) \mathbf{G}_I^h(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h) \\ &= \int_{\Gamma - \Gamma_{\mathbf{x}_{lm}}^h} v(s) \tilde{\mathbf{G}}(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h) + \int_{\Gamma_{\mathbf{x}_{lm}}^h} v(s) \tilde{\mathbf{G}}(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h) \\ &= \int_{\Gamma} v(s) \mathbf{G}(\mathbf{X}(s), \mathbf{x}_{lm}) ds + O(h \log h). \end{aligned}$$

In the derivation above, $\Gamma_{\mathbf{x}_{lm}}^h$ is the part of the interface that intersects with the ball $|\mathbf{X}_k - \mathbf{x}_{lm}| \leq Wh$, see the diagram Fig. 2. In the last two lines above, the second integral is bounded by $h \log h$ since the integrand may have a logarithm singularity and the integral interval is $O(h)$.

Theorem 3.7. *With the assumptions about the discrete delta function, the IB is first order convergent with a logarithm factor in the L^∞ norm*

$$|E_{lm}| \leq Ch \log h, \quad l, m = 1, 2, \dots, n-1. \quad (33)$$

Proof: Consider the error at a grid point E_{lm} , if \mathbf{x}_{lm} is close to the interface, that is,

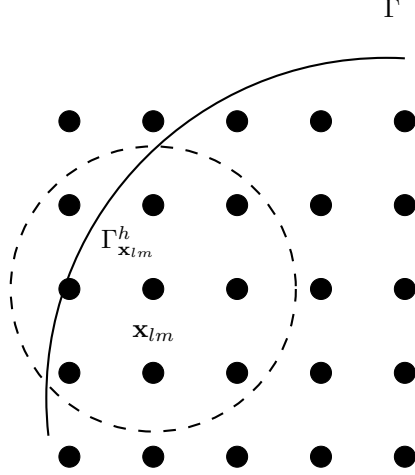


Figure 2: A diagram used in proving Lemma 3.6. The interface cuts through the ball centered at \mathbf{x}_{lm} with the radius Wh . Along the piece of the interface, the singular integral needs to be treated carefully.

$\text{dist}(\Gamma, \mathbf{x}_{lm}) \leq Wh$, we have

$$\begin{aligned}
E_{lm} &= ((A_h)^{-1} \mathbf{T}^{IB})_{kl} \\
&= ((A_h)^{-1} \mathbf{T}^{IB}_{reg})_{kl} + ((A_h)^{-1} \mathbf{T}^{IB}_{irr})_{lm} \\
&= O(h^2) + ((A_h)^{-1} \mathbf{T}^{IB}_{irr})_{lm} \\
&= \sum_{\text{dis}(\mathbf{x}_{ij}, \Gamma) \leq Wh} \left(h^2 T_{ij} (A_h)^{-1} \mathbf{e}_{ij} \frac{1}{h^2} \right)_{lm} + O(h^2) \\
&= \sum_{\text{dis}(\mathbf{x}_{ij}, \Gamma) \leq Wh} h^2 (\Delta_h u(x_i, y_j) - C_{ij}^{IB}) \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) + O(h^2) \\
&= \sum_{ij} h^2 \Delta_h u(x_i, y_j) \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) - \sum_{ij} h^2 C_{ij}^{IB} \mathbf{G}^h(\mathbf{x}_{ij}, \mathbf{x}_{lm}) + O(h^2) \\
&= \left(\int_{\Gamma} v(s) \tilde{\mathbf{G}}(\mathbf{X}(s), \mathbf{x}_{lm}) ds - \sum_k v_k \mathbf{G}_I^h(\mathbf{X}_k), \mathbf{x}_{lm}) \Delta s_k \right) + O(h) \\
&= O(h \log h),
\end{aligned}$$

after we apply Lemma 3.5 and Lemma 3.6. Note that, in the expansion of the summation from $\text{dis}(\mathbf{x}_{ij}, \Gamma) \leq Wh$ to all the interior grid points, we have used the fact that $\Delta_h u(x_i, y_j) = O(h^2)$ and $C_{ij}^{IB} = 0$ when $\text{dis}(\mathbf{x}_{ij}, \Gamma) > Wh$. If $\text{dist}(\Gamma, \mathbf{x}_{lm}) > Wh$, the proof above is still true except that we are not going to have a singular integration. Thus, we do not need the $\log h$ factor.

4 Conclusions and acknowledgments

We give a convergence proof of the immersed boundary (IB) method in the L^∞ norm. We show that the IB method is indeed first order accurate with a $\log h$ factor if a reasonable discrete delta function is used.

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